Non-Linear Transverse Vibrations of A Simply Supported Multi-Stepped Beam

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Abstract: In this study, the nonlinear vibrations of Euler-Bernoulli multiple-stepped beam are investigated. The beam is simply supported at both ends. The equations of motions are obtained using Hamilton's principle and made non-dimensional. The stretching effect induced non-linear terms to the equations. Forcing and damping terms are also included in the equations. A perturbation method is applied to the equations of motions. The first terms of the perturbation series led to the linear problem. Natural frequencies for the linear problem are calculated exactly for different step cases. Second order non-linear terms of the perturbation series behaved as corrections to the linear problem. Amplitude and phase modulation equations are obtained. Non-linear free and forced vibrations are investigated in detail. These analyses are repeated for different step ratios and step numbers.

Keywords: Nonlinear vibration, multi-stepped beam, perturbation method

Introduction

In real life, many engineering problems can be modeled as stepped beams such as bridges, rails, automotive industries, work pieces and machine elements. The most important aspect of vibration analysis is the calculation of natural frequencies. If the system is forced with a frequency close to its natural frequencies, the system comes to resonance state and the amplitudes increase dangerously. While computing the natural frequencies of the systems, assuming the systems to be linear makes the calculations easier but the results are usually not reliable. Because no system acts linearly obtained linear results may deceive us. Therefore, nonlinear effects originated from the stretching during the vibration of the beam should be included in the computations as well.

Many studies on beam vibrations, both linear and nonlinear, have previously been performed. The studies prior to 1979 are summarized by Nayfeh and Mook(1979). Particularly, the nonlinear behavior caused by the immobility of beam-ends has been analyzed by various researchers. Qaisi(1997) obtained the nonlinear vibration of beams with simply and clamped supports by using a power series approach and compared the results with existing solutions. Özkaya et al (1997) analyzed mass beam system for different boundary conditions. By considering the effects of stretching, they solved the obtained problem with the method of multiple scales, a perturbation technique. Özkaya(2002) considered a beam-masses system under simply supported end conditions. The effects of positions, magnitudes and number of the masses are investigated. For slightly curved beams with stretching, one may refer to Rehfield(1974) and Öz at al(1998).

Stepped beams are increasingly used in various branches of engineering, and so there are numerous studies on the vibration analysis of stepped beams with circular, rectangular cross sections and shafts. The first study on this subject is done by Taleb and Suppiger(1961). In their study, they obtained the frequency equation of a stepped beam with simple support and found the natural frequencies via the solution of the equation. Levinson (1976), on the other hand, listed the frequency equations for stepped beams with simple support but did not acquire any numeric results. Sato (1980) performed non-linear free vibration analysis for stepped beams with rectangular cross section and clamped and simple supports at both ends, and used the transfer matrix method for the solution. Balasubramanian and Subramanian (1985) analyzed vibrations for beams stepped at the middle. In another study, Balasubramanian et al (1990) acquired natural frequencies for high mode structures by using the study of Balasubramanian and Subramanian (1985). Jang and Bert (1989) obtained the frequency equation for stepped beam under various boundary conditions and computed the smallest natural frequencies for a circular cross-section beam. They compared the results with the results of Bert and Newberry (1986), who used a finite element analysis. In another study, Jang and Bert(1989) obtained natural frequencies for high mode structures using the frequency equation they acquired from the study by Jang and Bert (1989). Wang (1991) studied the vibration of stepped beams on elastic foundations. Rosa et al (1995) presented the free vibration analysis of stepped beams with intermediate elastic supports. Lee and Bergman (1994) submitted the vibration of stepped beams and rectangular plates. In their study, the structure with discontinues is divided into elemental substructures and the displacement field for each is obtained in terms of its dynamic Green's function. Aydın and Aksu(1981) used finite differences to estimate free vibration characteristics of regular changing beams and regular and irregular stepped beams and shafts. Energy functionality is minimized based on translocation elements and is computed as natural frequencies and mode forms. Krishnan at al(1998), studied the analysis of stepped beams using finite difference method and a single differential equation. In a study performed by Naguleswaran(2002), equations of motion of three different Euler-Bernoulli stepped beams with all states of boundary conditions are obtained and three natural frequencies are computed using the equations of motion. In another study, Naguleswaran(2002) considered three different types of stepped beams and investigated vibrations of a beam with up to three step changes. The dynamic stability of a stepped beam carrying mass is studied by Aldraihem and Baz(2002). The stepped beam equations of motion developed a discrete parameter form and a finite element form. Aydogdu and Taskin(2007) explored free vibration of simply supported FG beam and also they found the equations by applying Hamilton's principle. They used Navier type solution method in order to obtain frequencies. Kwon and Park(2002), focused on the effect of the position of the stepped point and thickness ratio on the dynamic characteristics of the system. The equation of motion and boundary equations are analytically obtained by using Hamilton's principle. The exact solutions are compared with the results obtained by FEM. Naguleswaran (2003) investigated the vibrations of beams with up to three step changes in cross section and axial force. The frequency equation for classical boundaries is expressed and the first three frequency parameters for the three types of beams are displayed. Kisa and Gurel(2007) represented the free vibration analysis of uniform and stepped cracked beams with circular cross sections. They used the finite element method and mode synthesis method and a non-linear problem separated into linear subsystems. Li (2001) analyzed the natural frequencies and mode shapes of multi-step beam and non-uniform beam with an arbitrary number of cracks and concentrated masses. Dong et al (205) investigated the natural frequencies and mode shapes of a stepped laminated composite Timoshenko beam. Their developed method can be used to deduce the frequency function of laminate stepped beams under other complex boundary conditions.

In this study, nonlinear vibration of an Euler-Bernoulli multi-stepped beam is considered. Natural frequencies are calculated for different locations, magnitudes and number of the steps. Nonlinear vibration analysis for multi-stepped beams is performed and the contributions of nonlinear terms on natural frequencies are investigated. Phase-modulation equations are acquired and frequency amplitude graphs are plotted using these equations.

Equation of Motion

For the system show in Fig. 1, the Lagrangian can be written as follows

$$£= \frac{1}{2}\sum_{m=0}^{n}\left[\int_{x_m^*}^{x_{m+1}^*}\rho A_{m+1}\,\dot{w}_{m+1}^{*\,2}\,dx^* - \int_{x_m^*}^{x_{m+1}^*} EI_{m+1} w''^{*\,2}_{m+1}\,dx^* - \int_{x_m^*}^{x_{m+1}^*} EA_{m+1}\left(u'^{*}_{m+1} + \frac{1}{2} w'^{*}_{m+1}{}^{2}\right)^2 dx^*\right]$$

(1)

$$x_0^* = 0, \qquad x_{n+1}^* = L$$

(2)

where L is the length, ρ is the density, A_{m+1} is cross sectional area of multi-stepped beam, E is Young's modulus, I_{m+1} is the moment of inertia of the multi-stepped beam's cross-section with respect to the neutral axis, n is number of steps, w is transverse displacement, ($\dot{}$) and ( )′ denote differentiations with respect to time t^* and the spatial variable x^* respectively.

The terms in Eq. (1) are the kinetics energies due to transverse motion, elastic energies due to bending and stretching of the beam, respectively.

Invoking Hamilton's principle,

$$\delta \int_{t_1}^{t_2} £\ dt^* = 0$$

(3)

and substituting the Lagrangian from Eq. (1), performing the necessary algebra and eliminating the axial displacements between equations, one finally obtained the following non-linear coupled integro-differential equations of motion:

$$\rho A_{m+1}\ddot{w}^*_{m+1} + EI_{m+1} w^{iv*}_{m+1} = \frac{EA_1}{2\sum_{r=0}^{n}(x^*_{r+1} - x^*_r)/\alpha_r^2}\left[\sum_{r=0}^{n}\int_{x_r^*}^{x_{r+1}^*} w'^{*2}_{r+1}\,dx^*\right] w''^*_{m+1} \qquad (m = 0,1,2....n)$$

(4)

There are n+1 equations in Eq. (4). In equation (4) $\alpha_r = d_{r+1}/d_1$ and $\alpha_o = 1$ (α_r is the ratio of r+1 th diameter to the first diameter). Note that viscous damping coefficient μ^*, external excitation with amplitude F^*_{m+1} and frequency Ω^* will be added to the equations. The boundary conditions can be written for this equation as follows

$$w_p^*(x_p,t^*) = w_{p+1}^*(x_p,t^*), \qquad w_p'^*(x_p,t^*) = w_{p+1}'^*(x_p,t^*),$$

$$EI_P\, w_p''^*(x_P,t^*) - EI_{P+1}\, w_{p+1}''^*(x_P,t^*) = 0; \quad EI_P\, w_P'''^*(x_P,t^*) - EI_{P+1}\, w_{p+1}'''^*(x_P,t^*) = 0$$

$$w_1^*(0,t^*) = w_1''^*(0,t^*) = 0\,, \quad w_{n+1}^*(L,t^*) = w_{n+1}''^*(L,t^*) = 0 \qquad \mathbf{p=1,2,3......n}$$

(5)

The equations and boundary conditions are made dimensionless using the following definitions

$$x = \frac{x^*}{L}, \qquad w_{m+1} = \frac{w^*_{m+1}}{R_{m+1}}, \qquad \eta_{m+1} = \frac{x_{m+1}}{L}, \qquad t = \frac{1}{L^2}\sqrt{\frac{EI_1}{\rho A_1}}\, t^*$$

(6)

where R_{m+1} is the radius of gyration of the stepped beam cross-section with respect to the neural axis. Substituting the dimensionless parameters into the equations of motion yield for the general case

$$\ddot{w}_{m+1} + \alpha_m^2 w^{iv}_{m+1} = \frac{1}{2\alpha_m^2\sum_{r=0}^{n}(\eta_{r+1}-\eta_r)/\alpha_r^2}\left[\sum_{r=0}^{n}\alpha_r^2\int_{\eta_r}^{\eta_{r+1}} w'^2_{r+1}\,dx\right] w''_{m+1} \qquad m=0,1,2.....n$$

(7)

and boundary conditions are

$$w_p(\eta_p,t) = \frac{\alpha_p}{\alpha_{p-1}} w_{p+1}(\eta_p,t) = 0\,, \qquad w_p'(\eta_p,t) = \frac{\alpha_p}{\alpha_{p-1}} w_{p+1}'(\eta_p,t)\,, \qquad w_p''(\eta_p,t) = \frac{\alpha_p^5}{\alpha_{p-1}^5} w_{p+1}''(\eta_p,t)\,,$$

$$w_p'''(\eta_p,t) = \frac{\alpha_p^5}{\alpha_{p-1}^5} w_{p+1}'''(\eta_p,t) \qquad w_1(0,t) = w_1''(0,t) = 0\,, \qquad w_{n+1}(1,t) = w_{n+1}''(1,t) = 0$$

p=1,2,3......n **(8)**

The equation of motion including damping and forcing is given below

$$\ddot{w}_{m+1} + \alpha_m^2 w^{iv}_{m+1} = \frac{1}{2\alpha_m^2\sum_{r=0}^{n}(\eta_{r+1}-\eta_r)/\alpha_r^2}\left[\sum_{r=0}^{n}\alpha_r^2\int_{\eta_r}^{\eta_{r+1}} w'^2_{r+1}\,dx\right] w''_{m+1} - 2\mu^*\,\dot{w}_{m+1} + F^*_{m+1}\cos\Omega t$$

m=0, 1, 2.....n

(9)

In equations (7 and 9) $\alpha_m = d_{m+1}/d_1$, $\alpha_o = 1$, $\eta_0=0$ and $\eta_{n+1}=1$.

Approximate Analytical Solution

In this section, approximate solutions of Eqs. (8) and (9) are searched with the boundary conditions. The method of multiple scales is applied to the partial differential equation systems and boundary conditions directly. Due to the absence of quadratic non-linearities, one can assume expansion of the form

$$w_{m+1}(x,t;\varepsilon) = \varepsilon\, w_{(m+1)1}(x,T_0,T_2) + \varepsilon^3\, w_{(m+1)3}(x,T_0,T_2) + ...$$

(10)

where ε is a small book-keeping parameter artificially inserted into the equations. This parameter can be taken 1 at the end upon keeping in mind, however, that deflections are small. We therefore investigated a weakly non-linear system. T_0=t and T_2=ε^2t are the fast and slow time scales. Let's consider only the primary resonance case and hence, the forcing and damping terms are ordered so that they counter the effect of non-linear terms

$$\mu^* = \varepsilon^2 \mu\,, \qquad F^*_{m+1} = \varepsilon^3\, F_{m+1}$$

(11)

the time derivatives are written as

$$(\dot{\ }) = D_0 + \varepsilon^2 D_2\,, \qquad (\ddot{\ }) = D_0^2 + 2\varepsilon^2 D_0 D_2\,, \qquad D_n = \frac{\partial}{\partial T_n}$$

(12)

Inserting Eqs. (10)-(12) into Eqs. (8) and (9), and equation coefficients of like powers of ε, one obtained

Order (ε):

$$D_0^2 w_{(m+1)1} + \alpha_m^2 w^{iv}_{(m+1)1} = 0$$

(13)

$$w_{p1}(\eta_p,t) = \frac{\alpha_p}{\alpha_{p-1}} w_{(p+1)1}(\eta_p,t)\,, \quad w'_{p1}(\eta_p,t) = \frac{\alpha_p}{\alpha_{p-1}} w'_{(p+1)1}(\eta_p,t)\,, \quad w''_{p1}(\eta_p,t) = \frac{\alpha_p^5}{\alpha_{p-1}^5} w''_{(p+1)1}(\eta_p,t)$$

$$w'''_{p1}(\eta_p,t) = \frac{\alpha_p^5}{\alpha_{p-1}^5} w'''_{(p+1)1}(\eta_p,t) \quad w_{11}(0,t) = w''_{11}(0,t) = 0\,, \quad w_{(n+1)1}(1,t) = w''_{(n+1)1}(1,t) = 0$$

(14)

Order (ε^3):

$$D_0^2 w_{(m+1)3} + \alpha_m^2 w^{iv}_{(m+1)3} = -2D_0 D_2 w_{(m+1)1} - 2\mu D_0 w_{(m+1)1} + \frac{1}{2\alpha_m^2 \sum_{r=0}^{n} (\eta_{r+1} - \eta_r)/\alpha_r^2}\left[\sum_{r=0}^{n} \alpha_r^2 \int_{\eta_r}^{\eta_{r+1}} w'^2_{(r+1)1} dx\right] w''_{(m+1)1} + F_{m+1}\cos \Omega T_0$$

(15)

$$w_{p3}(\eta_p,t) = \frac{\alpha_p}{\alpha_{p-1}} w_{(p+1)3}(\eta_p,t)\,, \quad w'_{p3}(\eta_p,t) = \frac{\alpha_p}{\alpha_{p-1}} w'_{(p+1)3}(\eta_p,t)\,, \quad w''_{p3}(\eta_p,t) = \frac{\alpha_p^5}{\alpha_{p-1}^5} w''_{(p+1)3}(\eta_p,t)\,,$$

$$w'''_{p3}(\eta_p,t) = \frac{\alpha_p^5}{\alpha_{p-1}^5} w'''_{(p+1)3}(\eta_p,t) \quad w_{13}(0,t) = w''_{13}(0,t) = 0\,,$$

$$w_{(n+1)3}(1,t) = w''_{(n+1)3}(1,t) = 0 \qquad \textbf{(16)}$$

3.1. Exact Solution To The Linear Problem

The linear problem is governed by Eqs. (13) and (14). Assuming solutions of the form

$$w_{(m+1)1} = [A(T_2)e^{i\omega T_0} + cc]Y_{m+1}(x)$$

(17)

where cc stands for complex conjugate of the preceding terms and substituting Eq. (17) into Eqs. (13) and (14), one obtains

$$Y^{iv}_{m+1} - \frac{1}{\alpha_m^2}\omega^2 Y_{m+1} = 0$$

(18)

$$Y_p(\eta_p) = \frac{\alpha_p}{\alpha_{p-1}} Y_{p+1}(\eta_p), \qquad Y'_p\ (\eta_p) = \frac{\alpha_p}{\alpha_{p-1}} Y'_{p+1}(\eta_p), \qquad Y''_p\ (\eta_p) = \frac{\alpha_p^5}{\alpha_{p-1}^5} Y''_{p+1}(\eta_p),$$

$$Y'''_p\ (\eta_p) = \frac{\alpha_p^5}{\alpha_{p-1}^5} Y'''_{p+1}(\eta_p), \qquad Y_1\ (0) = Y''_1\ (0) = 0, \qquad Y_{n+1}(1) = Y''_{n+1}(1) = 0$$

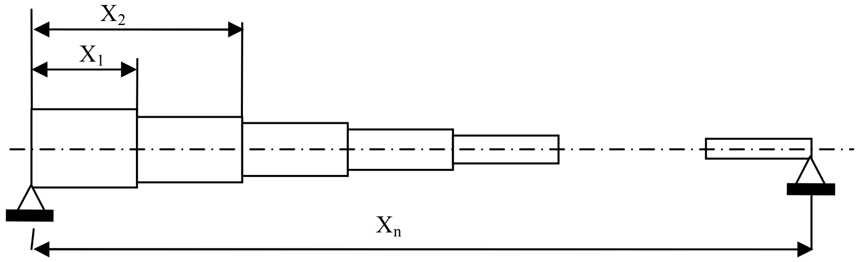
(19)

One can assume

$$Y_{m+1}(x) = C_{4m+1}\sin k_m\beta x + C_{4m+2}\cos k_m\beta x + C_{4m+3}\sinh k_m\beta x + C_{4m+4}\cosh k_m\beta x$$

(20)

for the solution of Eq. (18). Where $\beta = \sqrt{\omega}$ and $k_m = 1/\sqrt{\alpha_m}$. When the boundary and continuity conditions are applied to the equation of motion, frequency equations can be obtained. The multi-stepped beam system with simple end conditions is shown in the Fig.1.



$$w_1(0,t) = w''_1(0,t) = 0$$

$$w_{n+1}(L,t) = w''_{n+1}(L,t) = 0$$

Figure 1: A simply supported multi-stepped beam

The transcendental equation is numerically solved for the first three modes. The natural frequencies are listed for different α and η. Natural frequencies are given for one; two and three-step in Table (1-3).

α_1	η_1	ω_1	ω_2	ω_3	λ_1
0.5	0.2	4.76136	18.940256	45.241049	14.27798
	0.4	4.519872	23.954239	62.489572	9.10129
	0.6	5.154486	32.960370	59.994383	5.85852
	0.8	7.739925	28.929586	75.661069	2.86631
0.8	0.2	7.913373	32.063323	73.589218	3.70809
	0.4	8.140749	34.989506	77.524216	3.20758
	0.6	8.813577	35.901440	81.710293	2.38701
	0.8	9.639997	37.244195	83.454455	1.79947
1.0	0.5	9.869604	39.478417	88.826439	1.85055
2.0	0.2	15.479851	57.859172	151.322138	0.35828
	0.4	10.308972	65.920741	119.988766	0.73231
	0.6	9.039745	47.908479	124.979144	1.13766
	0.8	9.522719	37.880512	90.482097	1.78475
3.0	0.2	13.881543	74.798548	224.751695	0.28824
	0.4	7.831783	88.851217	136.674622	0.35699
	0.6	7.149344	46.586371	138.167797	0.58359
	0.8	8.747012	33.945355	85.829771	1.40013

Table- 1: The first three natural frequencies and the non-linear frequency correction coefficients of one-step beam for different step ratios and step locations

α_1	α_2	η_1	η_2	ω_1	ω_2	ω_3	λ_1
		0.1	0.3	5.953690	25.127266	59.138766	8.40157

0.8	0.6	0.1	0.5	6.323862	28.201730	59.967335	6.45813
		0.1	0.7	7.174081	27.950965	66.279061	4.28819
		0.1	0.9	7.854582	31.031939	68.938398	3.49167
		0.3	0.5	6.256803	29.719228	64.978786	6.27273
		0.3	0.7	7.196704	29.455923	71.398037	4.19713
		0.3	0.9	7.934874	32.638595	74.188639	3.29743
0.4	0.8	0.1	0.3	4.890838	23.486590	54.397537	9.90087
		0.1	0.5	3.714266	21.794759	46.113427	15.63241
		0.1	0.7	3.630831	15.493808	39.556262	24.40730
		0.1	0.9	3.875303	14.934684	32.864094	31.72566
		0.3	0.5	3.611960	23.322577	57.524918	11.02089
		0.3	0.7	3.265050	15.487205	57.356942	16.93990
		0.3	0.9	3.362330	15.177929	41.130527	19.45277
2.0	1.2	0.1	0.3	11.466482	47.998109	110.501310	0.84014
		0.1	0.5	12.000198	58.546396	108.929445	0.55186
		0.1	0.7	14.794350	56.059170	134.507356	0.31517
		0.1	0.9	18.596803	65.717636	136.873060	0.18292
		0.3	0.5	10.380079	41.728688	107.578383	0.56262
		0.3	0.7	11.193682	41.884783	140.201020	0.42031
		0.3	0.9	12.166286	57.264207	141.771501	0.51229
2.0	4.0	0.1	0.3	20.481951	111.238748	236.966957	0.11624
		0.1	0.5	17.494518	94.838302	211.264803	0.10834
		0.1	0.7	17.570170	69.802282	169.750578	0.15595
		0.1	0.9	18.800126	68.051795	145.611558	0.20869
		0.3	0.5	9.383658	104.081462	195.491493	0.30933
		0.3	0.7	11.159667	62.918320	167.058284	0.46987
		0.3	0.9	12.173198	59.990005	149.741932	0.58125

Table- 2: The first three natural frequencies and the non-linear frequency correction coefficients of two-step beam for different step ratios and step locations

α_1	α_2	α_3	η_1	η_2	η_3	ω_1	ω_2	ω_3	λ_1
0.8	0.6	0.3	0.1	0.3	0.5	2.672081	18.112328	41.765754	30.50196
			0.1	0.4	0.8	4.599781	19.681724	52.357778	13.61319
			0.2	0.5	0.7	3.421390	24.398166	50.468413	19.15705
			0.3	0.5	0.9	5.884991	25.159252	54.127598	5.92665
			0.5	0.7	0.9	6.558926	28.234674	60.753715	4.97262
2.0	1.0	0.8	0.1	0.3	0.5	7.564975	36.586408	86.878086	2.02849
			0.1	0.4	0.8	8.737297	43.147793	99.166220	1.01868
			0.2	0.5	0.7	8.069798	41.320096	80.532388	0.89315
			0.3	0.5	0.9	8.686695	38.685370	87.109404	0.68235
			0.5	0.7	0.9	8.669208	38.341971	88.505866	0.67547
2.0	4.0	6.0	0.1	0.3	0.5	18.078519	153.972235	271.133296	0.09026
			0.1	0.4	0.8	18.296694	112.443789	199.559828	0.10216
			0.2	0.5	0.7	12.185894	86.740381	260.992262	0.20382
			0.3	0.5	0.9	9.352935	103.998808	195.308334	0.30903
			0.5	0.7	0.9	7.798921	59.074062	192.978599	0.56567

Table-3: The first three natural frequencies and the non-linear frequency correction coefficients of three-step beam for different step ratios and step locations

Non-Linear Problem

Solving order ε^3, one obtains the non-linear corrections to the problem. Because the homogeneous Eqs. (13) and (14) have a non-trivial solution, the non-homogeneous problem (15) and (16) will have a solution only if a solvability condition is satisfied. To determine this condition, we firstly separated the secular and nonsecular terms by assuming a solution in the form of

$$w_{(m+1)3} = \phi_{m+1}(x,T_2)e^{i\omega T_0} + cc + W_{m+1}(x,T_0,T_2)$$

(21)

By substituting this solution into Eqs. (15) and (16), the terms producing secularities are eliminated. Hence the part of the equation determining $\phi_{(m+1)}$ is as follows:

$$\alpha_m^2\phi_{m+1}^{iv} - \omega^2\phi_{m+1} = -2i\omega(A' + \mu A)Y_{m+1} + \frac{3A^2\overline{A}}{2\alpha_m^2[\sum_{r=0}^{n}(\eta_{r+1}-\eta_r)/\alpha_r^2]}[\sum_{r=0}^{n}\alpha_r^2\int_{\eta_r}^{\eta_{r+1}}Y_{r+1}'^{\,2}dx]Y_{m+1}'' + \frac{1}{2}\mathrm{F}_{(m+1)}\mathrm{e}^{\mathrm{i}\sigma\mathrm{T}_2} \quad (22)$$

$$\phi_p = \frac{\alpha_p}{\alpha_{p-1}}\phi_{p+1} \quad , \quad \phi_p' = \frac{\alpha_p}{\alpha_{p-1}}\phi_{p+1}', \quad , \quad \phi_p'' = \frac{\alpha_p^5}{\alpha_{p-1}^5}\phi_{p+1}'' \quad , \quad \phi_p''' = \frac{\alpha_p^5}{\alpha_{p-1}^5}\phi_{p+1}'''$$

$$\phi_1(0) = \phi_1''(0) = 0\,, \qquad \phi_{n+1}(1) = \phi_{n+1}''(1) = 0$$

(23)

In obtaining these equations, one emplayes the first order solutions (17). One can also assume that the external excitation frequency is close to one of the natural frequencies of the system; that is,

$$\Omega = \omega + \varepsilon^2\sigma$$

(24)

where σ is a detuning parameter of order 1. After some algebraic manipulations, one can obtain the solvability condition for Eqs. (22) and (23) as

$$2i\omega(A' + \mu A) + \frac{3}{2\sum_{r=0}^{n}(\eta_{r+1}-\eta_r)/\alpha_r^2}A^2\overline{A}b^2 - \frac{1}{2}fe^{i\sigma T_2} = 0$$

(25)

where the equations are normalized by requiring

$$\sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}\alpha_r^4\mathrm{Y}_{r+1}^2\mathrm{dx} = 1, \quad \sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}\alpha_r^2\mathrm{Y}_{r+1}'^2\mathrm{dx} = \mathrm{b}, \quad \sum_{r=0}^{n}\int_{\eta_r}^{\eta_{r+1}}\alpha_r^4\mathrm{F}_{r+1}\mathrm{Y}_{r+1}\mathrm{dx} = \mathrm{f}$$

(26)

The complex amplitude A can be written in terms of a real amplitude a and a phase θ

$$A = \frac{1}{2}a(T_2)e^{i\theta T_2}$$

(27)

Substituting Eq. (27) into Eq. (25), and separating real and imaginary parts, one obtained finally phase and modulation equations

$$\omega a\gamma' = \omega a\sigma - \frac{3}{16}b^2a^3\Lambda + \frac{1}{2}f\cos\gamma\,, \qquad \omega a' = -\omega\mu a + \frac{1}{2}f\sin\gamma$$

(28)

where Λ and γ are defined as

$$\gamma = \sigma T_2 - \theta \qquad \Lambda = 1/\sum_{r=0}^{n}[\frac{(\eta_{r+1}-\eta_r)}{\alpha_r^2}]$$

(29)

In this section amplitude and phase modulation equations are determined from the non-linear analysis for multiple stepped.

Numerical Results

In this section numerical examples are presented for different step numbers. Firstly, the linear natural frequencies for different step numbers (n=1,2,3) for various α and η values are found and given in Tables 1-3. As long as the beam supports are fixed nonlinearity is actually negligible though it has some cubic order of perturbation. This effect which is well known as slenderness parameter is considered in the numerical results presented. When the step number is increased, the natural frequency value decreased for diminishing step ratios. The decrease is inclined to the value of cone's natural frequency. When $\eta = 1$, the natural frequency values are obtained as straight simple supported beam. Also, the linear natural frequencies for various step ratios are compared with those given by Naguleswaran (2002) and are observed similar results.

Then, the non-linear frequencies for free undamped vibrations are calculated similarly. In equation (28), by taking $\mu=f=\sigma=0$, one obtains

$a'=0$ and $a = a_0$ (constant)
(30)

Note that a_0 is the steady-state real amplitude of response. Hence the non-linear frequency is

$$\omega_{nl} = \omega + \theta' = \omega + \lambda a_0^{\,2} \qquad (31)$$

where

$$\lambda = \frac{3}{16}\frac{\Lambda b^2}{\omega}$$
(32)

In this order of approximation, thus, the non-linear frequencies had a parabolic relation with respect to the maximum amplitude of vibration. λ could be defined as the non-linear correction coefficient. For different α and η, the nonlinear correction coefficients are listed in Tables 1-3 for the first mode for different step numbers. λ is a measure of the stretching effect. The non-linearities are of hardening type. When the stepped ratio is increased, the nonlinear frequency correction coefficient decreased for one step case. Similarly, as the step location changed from left to right, the stretching effects decreased regardless of the step ratios.

The curves showing the relationships between nonlinear frequency and amplitude are given in Figures 2-4 for different α, η values and different step numbers. In figures 2, non-linear frequency-amplitude curve is drawn for one step case and different step ratios. In figure-2, as η increased, the effects of stretching decreased. In figure 3, non-linear frequency versus amplitude is plotted for two step case only when η_1=0.3 and η_2=0.5, 0.7, 0.9. For α_1=2.0 and α_2=4.0, as the stepped location (η_2) increased, the stretching effects increased. Figs. 4 show non-linear frequency versus amplitude for three step case for the first mode only when η_1-η_2-η_3: 0.1-0.3-0.5, 0.2-0.5-0.7, and 0.3-0.5-0.9. For α_1=0.8, α_2=0.6 and α_3=0.3, as the step location shifted from left to right, the stretching effects decreased. For all step cases, the stretching effects decreased as step ratios increased. The results for one step, two steps and three steps are given in Figure 5-7 for different step parameters.

One now can consider damping and external excitation case. In Eq. (28), when the system reaches the steady state region, a' and γ' vanish and hence one obtains the following equations.

$$\sigma = \frac{3}{16}\frac{a^2 b^2 \Lambda}{\omega} \mp \sqrt{\frac{f^2}{4\omega^2 a^2} - \mu^2}$$
(33)

The detuning parameter shows the nearness of the external excitation frequency to the natural frequency of system. Several figures can be drawn using Eq. (33) assuming f=1 and damping coefficient μ=0.2. Frequency response curves are presented in Figs. 8-11. In Figs. 8-9, the frequency–response curves for one step case are shown when η_1=0.2, 0.4, 0.6, 0.8. In Fig. 8, when η_1 decreased and provided that other parameters are kept constant, multi-valued regions increased drastically (α_1=0.5). In Fig. 9, the effect of forcing is maximum when η_1=0.6 and, is minimum when η_1=0.2(α_1=3.0). Fig. 10 shows frequency–response curves for two steps case for the first mode only when η_1=0.1 and η_2=0.3, 0.5, 0.7, 0.9. When the step position (η_2) is shifted from left to right, the amplitudes decreased (α_1=0.8 and α_2=0.6). Fig. 11 shows frequency–response curves for three steps case for the first mode only when η_1-η_2-η_3: 0.1-0.3-0.5, 0.2-0.5-0.7, and 0.3-0.5-0.9. The effect of forcing is maximum η_1=0.3, η_2=0.5 or η_3 = 0.9, is minimum when η_1=0.1, η_2=0.3 or η_3=0.5. Similar conclusions can be drawn. The effect of stretching bends the curves to the right causing multi-valued regions for the solution. This phenomenon is the well-known jump phenomena.

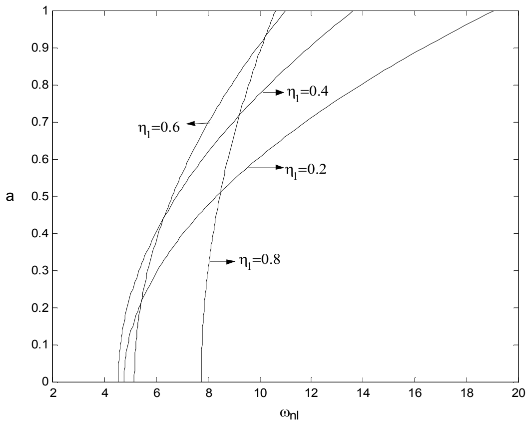


Figure 2: Non-linear frequency versus amplitude for different step location values (first mode, one step α_1=0.5)

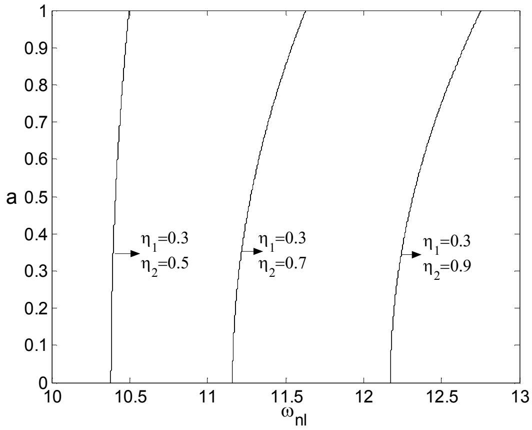


Figure 3: Non-linear frequency versus amplitude for different step location values(first mode; two step α_1=2.0, α_2=4.0)

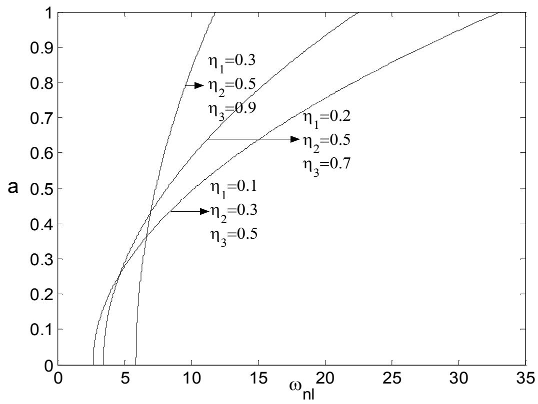


Figure 4:Non-linear frequency versus amplitude for different step location values(first mode, three step α_1=0.8,α_2=0.6,α_3=0.3)

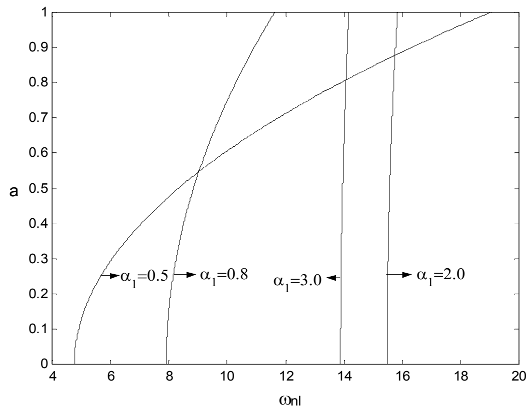


Figure 5: Non-linear frequency versus amplitude for different step ratio values (first mode, one step, η_1=0.2)

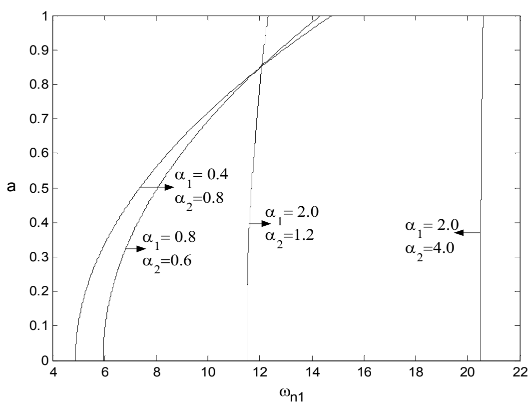


Figure 6: Non-linear frequency versus amplitude for different step ration values (first mode, two step η_1=0.1, η_2=0.3)

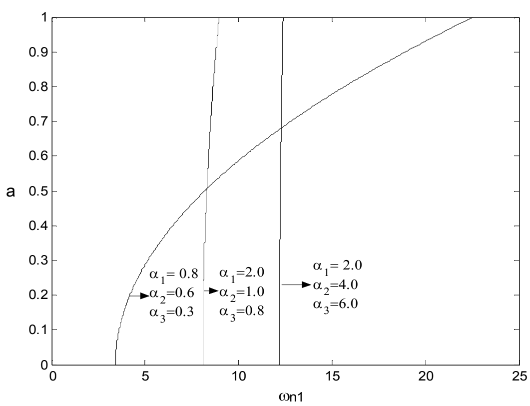
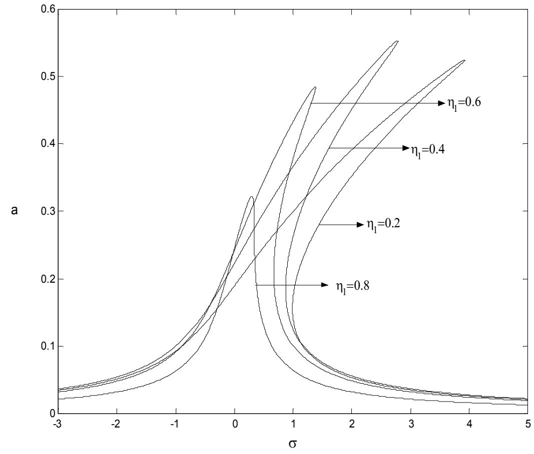


Figure 7: Non-linear frequency versus amplitude for different step ration values (first mode, three step η_1=0.2, η_2=0.5, η_3=0.7)



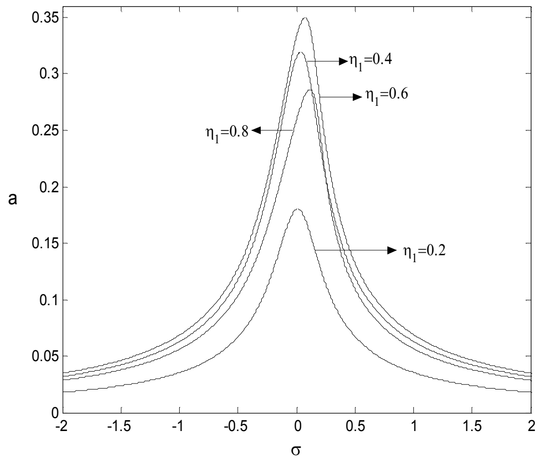
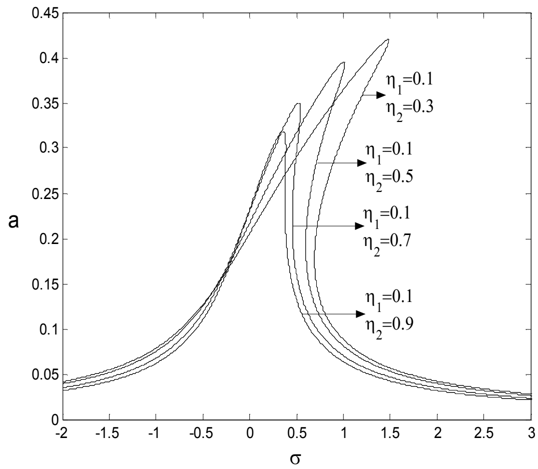


Figure 8: Frequency-response curves for different step locations (first mode, one step, α=0.5)

Figure 9: Frequency-response curves for different step locations (first mode, one step, α_1=3.0)



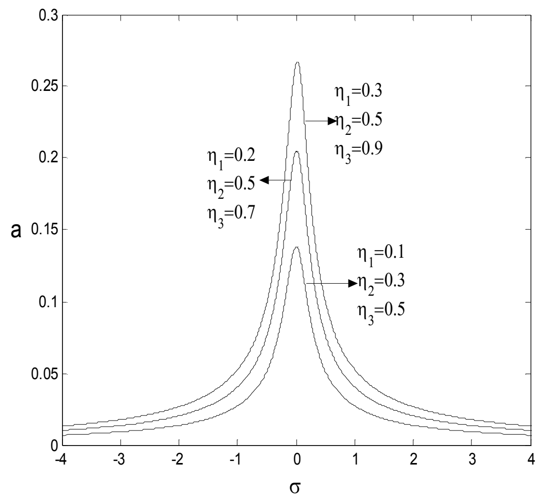


Figure 10: Frequency-response curves for different step locations (first mode, two step α_1=0.8 α_2=0.6)

Figure 11: Frequency-response curves for different step locations (first mode, three step α_1=2.0, α_2=4.0, α_3=6.0)

Concluding Remarks

The non-linear response of multi-stepped beam is investigated. The beam is simply supported at both ends. The non-linear equations of motion including stretching due to immovable end conditions are derived. Forcing and damping terms are added to the equations. Linear and non-linear analyses are performed. Approximate solutions are searched by applying the method of multiple scales directly to the partial differential equations. The first term led to the linear problem. When the boundary conditions and continuity are applied to the equation of motion, frequency equations are obtained and given for one step beam. Mode shapes and natural frequencies are calculated for different step ratios, step locations and number of the steps. The second terms provided the non-linear corrections to the linear problem. Non-linear frequency-amplitude and forcing frequency-amplitude relations are investigated and plotted. For one step beam, when the step number is increased, the natural frequency value decreased for diminishing step ratios. The decrease is inclined to the value of cone's natural frequency. As the step ratio is increased, the natural frequencies and nonlinear frequencies generally increased, but after the step ratio value 2, we observed a decreasing trend in nonlinear frequencies. One can observe that the stretching caused a non-linearity of the hardening type. When the step ratio is increased (α), the effect of stretching on the non-linear frequencies generally decreased. For forced and damped vibrations, since the non-linearity is of hardening type, the frequency-response curves bent to the right, causing an increase in the multi-valued regions for the solution.

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